

ON THE INVARIANTS OF QUADRATIC DIFFERENTIAL FORMS*

BY

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In the following paper I propose to investigate, by means of LIE'S theory of continuous groups, the problem of determining the number of invariants of the general quadratic differential form in n variables.

§ 1. *Introduction.*

The n -ary quadratic differential forms

$$(1) \quad \phi \equiv \sum_{i=1}^n \sum_{k=1}^i a_{ik}(x_1 \cdots x_n) dx_i dx_k \quad (a_{ik} \equiv a_{ki}),$$

the invariants of which are to be considered, will be subjected to the restriction that their discriminants, the determinants

$$(2) \quad a \equiv |a_{11} \cdots a_{nn}|,$$

shall not vanish identically. The coefficients a_{ik} , together with their partial derivatives of all orders, will be continuous functions of the n independent variables $x_1 \cdots x_n$.

These forms will be subjected to the infinite group of all point transformations in the variables $x_1 \cdots x_n$, which are generated by the infinitesimal transformation

$$(3) \quad Xf \equiv \sum_{r=1}^n \xi_r(x_1 \cdots x_n) \frac{\partial f}{\partial x_r}.$$

The quantities ξ_r , with their partial derivatives of all orders, will be supposed to be continuous functions of the n independent variables $x_1 \cdots x_n$, but, except for this restriction, wholly arbitrary. This infinite group of transformations includes all point transformations in the neighborhood of the identical transformation

$$x'_i = x_i \quad (i = 1, 2, \cdots, n).$$

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Under a transformation

$$(4) \quad x'_i = x'_i(x_1 \cdots x_n) \quad (i=1, 2, \cdots, n),$$

the form ϕ becomes

$$(5) \quad \phi' \equiv \sum_{i=1}^n \sum_{k=1}^n a'_{ik}(x'_1 \cdots x'_n) dx'_i dx'_k \quad (a'_{ki} \equiv a'_{ik}),$$

where

$$(6) \quad a'_{ik} = \sum_{r=1}^n \sum_{s=1}^n a_{rs} \frac{\partial x_i}{\partial x'_r} \frac{\partial x_k}{\partial x'_s} \quad (i, k=1, 2, \cdots, n).$$

An *invariant* of the form ϕ under the above group is a function f of the variables x , of the coefficients a_{ik} and of their derivatives of various orders, taken with respect to the variables x , such that, whether it be constructed from the original variables x and the coefficients and derivatives from the original form ϕ , or from the variables x' and the coefficients and derivatives from the transformed form ϕ' , its value is the same whatever transformation of the group is employed. (It will appear in the course of the work that the variables x cannot enter explicitly.)

The *order* of an invariant is the order of the highest derivative appearing in it.

The problem of the present paper is that of the determination of the number of invariants of each order μ for the general form in n variables.

It will be shown that such a form has

$$(7) \quad J_{n2} = \frac{(n-2)(n-1)n(n+3)}{12}$$

invariants of order two, and

$$(8) \quad I_{n\mu} = n \frac{\mu-1}{2} \frac{(n+\mu-1)!}{(n-2)! (\mu+1)!}$$

invariants of order $\mu > 2$, provided $n \geq 3$. The latter formula holds also when $\mu > 3$, if $n = 2$.

The cases $\mu = 0$, $\mu = 1$ have been treated by RICCI,* who showed that no invariants of these orders exist.

The case $n = 2$ has been fully studied by ŻORAWSKI,† who showed that there is in this case one invariant of order two, one of order three, and $\mu - 1$ of order $\mu > 3$.

LEVI-CIVITA‡ has found a lower limit for the number of invariants of a given order for the general form. He expressly states,§ however, that he does

* RICCI, *Annali di Matematica*, ser. 2, vol. 14 (1886), pp. 4-5.

† ŻORAWSKI, *Acta Mathematica*, vol. 16 (1892), p. 41.

‡ LEVI-CIVITA, *Reale Istituto Veneto, Atti*, ser. 7, vol. 5₂ (1894), pp. 1468, 1507.

§ *Ibid.*, p. 1507.

not determine the exact number. It is to this determination that the present paper is devoted.

ZORAWSKI and LEVI-CIVITA have used in their work the methods of LIE. By these methods the invariants of the form are all to be found as solutions of complete systems of homogeneous linear partial differential equations of the first order. Hence the number of independent invariants is equal to the number of independent solutions of the complete systems, and this, in turn, is equal to the excess of the number of variables in the equations over the number of independent equations. Hence the entire determination of the number of independent invariants of the form reduces to the determination of the number of independent equations in the complete system which the invariants must satisfy. It is precisely this determination which is omitted from the work of LEVI-CIVITA.

The direct method of finding the number of these independent equations is the computation of determinants from the matrix of the system. The great number of the equations and their somewhat complicated form renders this method, however, impracticable. Hence it is necessary to have recourse to a special method particularly adapted to the type of equations under consideration. This method is, in outline, the following. By means of a simple lemma the problem is reduced to the determination of the independence of the equations of two different sets. Each of these determinations can be carried on by methods of mathematical induction. By suitable changes of the variables and by the use of certain particular forms ϕ the work is materially simplified.

§ 2. *Differential equations of the problem.*

In order to determine by LIE's methods the invariants of order μ we must first "extend" the group Xf to the coefficients a_{ik} of the form and to their derivatives of order μ and lower.* To do this we note that, under the extended transformation, ϕ does not change its value, and consequently, if we denote by X_0f the transformation Xf extended to the coefficients a_{ik} , we have

$$(1) \quad X_0\phi \equiv \sum_{i=1}^n \sum_{k=1}^n \left\{ Xa_{ik} + \sum_{r=1}^n \left(a_{kr} \frac{\partial \xi_r}{\partial x_i} + a_{ir} \frac{\partial \xi_r}{\partial x_k} \right) \right\} dx_i dx_k \equiv 0,$$

the identity having reference to the n differentials dx_1, \dots, dx_n .

Hence we have†

$$(2) \quad Xa_{ik} + \sum_{r=1}^n \left(a_{kr} \frac{\partial \xi_r}{\partial x_i} + a_{ir} \frac{\partial \xi_r}{\partial x_k} \right) = 0.$$

* Cf. LIE, *Mathematische Annalen*, vol. 24 (1884), p. 575. ZORAWSKI, *Acta Mathematica*, vol. 16 (1892), p. 9.

† Cf. KILLING, *Crelle's Journal*, vol. 109 (1891), p. 121.

From this the extension to the derivatives is obtained by successive applications of the formula

$$(3) \quad X \left(\frac{\partial^{m+1} a_{ik}}{\partial x_{l_1} \cdots \partial x_{l_m} \partial x_{l_{m+1}}} \right) = \frac{\partial}{\partial x_{l_{m+1}}} \left\{ X \left(\frac{\partial^m a_{ik}}{\partial x_{l_1} \cdots \partial x_{l_m}} \right) \right\} - \sum_{r=1}^n \frac{\partial}{\partial x_r} \left(\frac{\partial^m a_{ik}}{\partial x_{l_1} \cdots \partial x_{l_m}} \right) \frac{\partial \xi_r}{\partial x_{l_{m+1}}} . *$$

We shall use the following notations :

$$(4) \quad \begin{aligned} a_{ik|l_1 \cdots l_\mu} &\equiv \frac{\partial^\mu a_{ik}}{\partial x_{l_1} \cdots \partial x_{l_\mu}}, & a_{ik|l_1 \cdots l_\mu} &\equiv X_{l_1 \cdots l_\mu} a_{ik}, \\ p_{ik|l_1 \cdots l_\mu} &\equiv \frac{\partial f}{\partial \left(\frac{a_{ik}}{a_{ik|l_1 \cdots l_\mu}} \right)}, & \xi_r|l_1 \cdots l_\mu &\equiv \frac{\partial^\mu \xi_r}{\partial x_{l_1} \cdots \partial x_{l_\mu}}, \end{aligned}$$

and shall say that each of these quantities is of order μ . For convenient use in the text, however, the notation (4) will sometimes be replaced respectively by the notations :

$$(4') \quad a_{ik|l_1 \cdots l_\mu}, \quad a_{ik|l_1 \cdots l_\mu}, \quad p_{ik|l_1 \cdots l_\mu}, \quad \xi_r|l_1 \cdots l_\mu.$$

These notations differ somewhat from those used by LIE and ŻORAWSKI. They are, however, essentially the same as those of LEVI-CIVITA and have the advantage that they make obvious certain relations which hold when for the derivatives of the coefficients a_{ik} are substituted the well-known symbols of CHRISTOFFEL.

The invariants of order μ or lower are all solutions of the linear partial differential equations obtained by equating to zero the coefficients of the quantities ξ and their derivatives in the expression

$$(5) \quad \begin{aligned} \bar{X}f &\equiv Xf + \sum_{i=1}^n \sum_{k=1}^i a_{ik} p_{ik} + \sum_{i=1}^n \sum_{k=1}^i \sum_{l_1=1}^n a_{ik} p_{i l_1 l_1} + \cdots \\ &+ \sum_{i=1}^n \sum_{k=1}^i \sum_{l_1=1}^n \sum_{l_2=1}^{l_1} \cdots \sum_{l_{\mu-1}=1}^{l_{\mu-2}} a_{ik} p_{i l_1 \cdots l_{\mu-1} l_{\mu-1}}. \end{aligned}$$

These equations form a complete system. † The coefficients of the quantities ξ give

$$(6) \quad \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0.$$

That is, the invariants cannot contain the variables x explicitly.

* LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 1, p. 545.

† LIE, *Mathematische Annalen*, vol. 24 (1884), p. 566.

It is easily seen that the system remains complete after these equations are stricken out. We have, therefore, to discuss the complete system obtained by equating to zero the coefficients of the ξ 's and of their derivatives in the expression

$$(7) \quad \begin{aligned} X_\mu f \equiv & \sum_{i=1}^n \sum_{k=1}^i a_{ik} p_{ik} + \sum_{i=1}^n \sum_{k=1}^i \sum_{l_1=1}^n a_{ik} p_{ik} \sum_{l_1}^{l_1} + \cdots \\ & + \sum_{i=1}^n \sum_{k=1}^i \sum_{l_1=1}^n \sum_{l_2=1}^{l_1} \cdots \sum_{l_{\mu-1}=1}^{l_{\mu-2}} a_{ik} p_{ik} \sum_{l_1 \cdots l_{\mu-1}}^{l_{\mu-1}}. \end{aligned}$$

We shall call the equations of this system 'the equations of order μ .'

We find

$$(8) \quad a_{ik} = - \sum_{r=1}^n (a_{ir} \xi_r + a_{kr} \xi_r),$$

$$(9) \quad a_{ik}^{l_1} = - \sum_{r=1}^n \left\{ (a_{ir} \xi_r + a_{kr} \xi_r + a_{ik} \xi_r) + (a_{ir} \xi_r + a_{kr} \xi_r) \right\},$$

$$(10) \quad \begin{aligned} a_{ik}^{l_1 l_2} = & - \sum_{r=1}^n \left\{ (a_{ir} \xi_r + a_{kr} \xi_r + a_{ik} \xi_r + a_{ik} \xi_r) \right. \\ & + (a_{ir} \xi_r + a_{kr} \xi_r + a_{kr} \xi_r + a_{kr} \xi_r + a_{ik} \xi_r) \\ & \left. + (a_{ir} \xi_r + a_{kr} \xi_r) \right\}, \end{aligned}$$

etc.

The formulæ giving the values of the a 's of higher orders are important only so far as the terms containing the highest derivatives of the ξ 's are concerned. These are easily seen to be, for $a_{ik|l_1 \cdots l_n}$,

$$(11) \quad - \sum_{r=1}^n (a_{ir} \xi_r + a_{kr} \xi_r).$$

The above expressions for $X_\mu f$ and the a 's make evident that peculiarity of the equations determining the invariant which renders possible the use of the special method of the present paper in determining their independence. For it is easily seen that, since the a 's of order $\mu + 1$ contain derivatives of the ξ 's of all orders $1, 2, \cdots, \mu + 2$, while the a 's of order μ contain derivatives of the ξ 's of orders $1, 2, \cdots, \mu + 1$ only, the equations of order $\mu + 1$ may be obtained by annexing certain terms to the equations of order μ , and by adding certain equations to the system. The annexed terms and the added equations come from the expression

$$(12) \quad X_{\mu+1} f - X_\mu f \equiv \sum_{i=1}^n \sum_{k=1}^i \sum_{l_1=1}^n \sum_{l_2=1}^{l_1} \cdots \sum_{l_{\mu-1}=1}^{l_{\mu-2}} \sum_{l_\mu=1}^{l_{\mu-1}} a_{ik} p_{ik} \sum_{l_1 \cdots l_{\mu+1}}^{l_{\mu+1}},$$

and hence contain no p 's of order less than $\mu + 1$, *i. e.*, they contain none of the p 's which appear in the equations of order μ or lower.

The added equations we shall call the *final* equations of order μ .

The equations of order μ which are obtained by annexing terms to the equations of order zero we call the *primary* equations of order μ . The remaining equations of order μ we call the *secondary* equations of that order.

§ 3. *Decomposition of the problem.*

The form of the equations under discussion permits us to decompose the problem of determining their independence into two auxiliary problems. The possibility of this decomposition is a consequence of the following

LEMMA.—*If all the equations of order $\mu - 1$ are independent, and if the final equations of order μ are all independent, then all the equations of order μ are independent.*

For suppose there are M equations of order $\mu - 1$ and N final equations of order μ . Then there are $M + N$ equations of order μ . At least one determinant, Δ_1 , of order M , formed from the matrix of the equations of order $\mu - 1$, does not vanish; and at least one determinant, Δ_2 , of order N , formed from the matrix of the final equations of order μ does not vanish. Form now that determinant, Δ_3 , of order $M + N$, the first M columns of which contain in their first M rows the elements of Δ_1 , and the last N columns of which contain in their last N rows the elements of Δ_2 . Then, since the final equations of order μ contain no p 's of order lower than μ , the elements in the first M columns and last N rows of Δ_3 are all zero. Hence

$$(1) \qquad \Delta_3 = \Delta_1 \Delta_2 \neq 0,$$

and consequently, since at least one determinant of order $M + N$ formed from the matrix of the equations of order $\mu + 1$ does not vanish, these equations are all independent.

This theorem was used by ŻORAWSKI* for the simple case of the binary forms. It allows us to resolve the problem under discussion into two parts, *viz.*:

I. *The determination of the independence of the final equations of general order μ .*

II. *The determination of the existence of an order μ for which all the equations are independent.*

It will appear that for $n \geq 3$ the equations of order two are all independent. For $n = 2$ this is not true, but the equations of order three are all independent.

* ŻORAWSKI, *Acta Mathematica*, vol. 16 (1892), p. 21.

§ 4. Independence of the final equations.

We shall first show that if the final equations of order $\mu + 1$ are not independent the same is true of the final equations of order μ .

The final equations of order $\mu + 1$ are readily seen to be obtained by equating to zero the coefficient of the $(\mu + 2)$ th derivatives of the ξ 's in the expression

$$(1) \quad \Gamma_{\mu+1} f = \sum_{i,k} \sum_{l_1, l_2, \dots, l_{\mu+1}} p_{ik|l_1 l_2 \dots l_{\mu+1}} \sum_r \left(a_{ir|kl_1 \dots l_{\mu+1}} \xi_r + a_{kr|i l_1 \dots l_{\mu+1}} \xi_r \right) \\ (i, k, l_1, \dots, l_{\mu+1} = 1, 2, \dots, n; k \leq i, l_1 \geq l_2 \geq \dots \geq l_{\mu} \geq l_{\mu+1}).$$

If these equations are dependent there exists a set of multipliers, not all zero, such that, if they be applied to the corresponding equations and the results added, the coefficient of each p is identically zero. But since, in forming the equations, we collect the coefficients of the various quantities $\xi_r|_{i l_1 \dots l_{\mu+1}}$, and equate them to zero, it follows that if we consider these quantities, not as derivatives of the n ξ 's, but as undetermined multipliers, then the necessary and sufficient condition that the final equations of order $\mu + 1$ shall not be all independent is that there shall exist a set of quantities $\xi_r|_{i l_1 \dots l_{\mu+1}}$, not all zero, such that they make the expression $\Gamma_{\mu+1} f$, considered as a polynomial in the variables $p_{ik|l_1 l_2 \dots l_{\mu+1}}$, vanish identically. That is, there must exist a set of quantities $\xi_r|_{i l_1 \dots l_{\mu+1}}$, not all identically zero, satisfying the equations

$$(2) \quad \sum_{r=1}^n \left(a_{ir|kl_1 \dots l_{\mu+1}} \xi_r + a_{kr|i l_1 \dots l_{\mu+1}} \xi_r \right) = 0 \\ (i, k, l_1, \dots, l_{\mu+1} = 1, 2, \dots, n; k \leq i, l_1 \geq l_2 \geq \dots \geq l_{\mu} \geq l_{\mu+1}).$$

Suppose that such a set of multipliers exists. There is, therefore, at least one index λ , appearing as the last index $l_{\mu+1}$ for which the corresponding equations are satisfied by multipliers $\xi_r|_{i l_1 \dots l_{\mu+1}}$, not all zero. The ξ 's appearing in the other equations may or may not vanish.

Now it amounts merely to a renaming of the x 's if we take x_{λ} as x_1 . Hence there is a set of equations:

$$(3) \quad \sum_{r=1}^n \left(a_{ir|kl_1 \dots l_{\mu 1}} \xi_r + a_{kr|i l_1 \dots l_{\mu 1}} \xi_r \right) = 0 \\ (i, k, l_1, \dots, l_{\mu} = 1, 2, \dots, n; k \leq i, l_1 \geq l_2 \geq \dots \geq l_{\mu}),$$

satisfied by a set of quantities $\xi_r|_{i l_1 \dots l_{\mu 1}}$, not all of which are zero. But then, putting

$$(4) \quad \eta_r|_{i l_1 \dots l_{\mu}} = \xi_r|_{i l_1 \dots l_{\mu 1}},$$

we have

$$(5) \quad \sum_{r=1}^n \left(a_{ir|kl_1 \dots l_{\mu}} \eta_r + a_{kr|i l_1 \dots l_{\mu}} \eta_r \right) = 0 \\ (i, k, l_1, \dots, l_{\mu} = 1, 2, \dots, n; k \leq i, l_1 \geq l_2 \geq \dots \geq l_{\mu}).$$

These are, however, precisely the equations which, if satisfied by a set of η 's not all zero, give the necessary and sufficient condition that the final equations of order μ shall not be all independent. Hence we have the following

THEOREM: *If the final equations of order $\mu + 1$ are not all independent, the final equations of order μ are not all independent.*

COROLLARY: *If the final equations of order $\mu > 1$ are not all independent, the final equations of order unity are not all independent.*

It may now be shown by means of a device already used for a similar purpose by RICCI,* though in connection with finite rather than infinitesimal transformations, that the final equations of order unity are all independent. This device consists in substituting for the derivatives, $a_{ik|l}$, of the coefficients of the form, the well-known three-index symbols

$$(6) \quad \left[\begin{matrix} ik \\ l \end{matrix} \right] \equiv \frac{1}{2} (a_{il} + a_{kl} - a_{ik})$$

of CHRISTOFFEL. The equations obtained from the final equations of order one by means of this change of variables are easily shown to be independent, and hence the original equations must have been so.

The three-index symbols satisfy the relations

$$(7) \quad \left[\begin{matrix} ik \\ l \end{matrix} \right] = \left[\begin{matrix} ki \\ l \end{matrix} \right], \quad a_{ik} = \left[\begin{matrix} il \\ k \end{matrix} \right] + \left[\begin{matrix} kl \\ i \end{matrix} \right].$$

Hence there are $n^2(n+1)/2$ distinct symbols, and they can replace the system of the $n^2(n+1)/2$ first derivatives, $a_{ik|l}$, of the coefficients of the form.

Putting

$$(8) \quad q_{ik} \equiv \frac{\partial f}{\partial \left[\begin{matrix} ik \\ l \end{matrix} \right]},$$

we find that

$$(9) \quad \Gamma_1 f \equiv \sum_{i=1}^n \sum_{k=1}^i \sum_{l=1}^n p_{ik} \sum_{r=1}^n (a_{ir} \xi_r + a_{kr} \xi_r)$$

becomes

$$(10) \quad \Gamma_1 f \equiv \sum_{i=1}^n \sum_{k=1}^i \sum_{l=1}^n q_{ik} \sum_{r=1}^n a_{ir} \xi_r \equiv \sum_{i=1}^n \sum_{k=1}^i \sum_{r=1}^n \xi_r \sum_{l=1}^n a_{ir} q_{ik},$$

and hence the final equations of order unity are transformed into

$$(11) \quad \sum_{l=1}^n a_{ir} q_{ik} = 0 \quad (i, k, r = 1, 2, \dots, n; k \leq i).$$

* RICCI, *Annali di Matematica*, ser. 5, vol. 14 (1886), p. 5.

These $n^2(n+1)/2$ equations contain $n^2(n+1)/2$ variables q_{ik} , and hence, as their determinant is

$$(12) \quad a^{\frac{n(n+1)}{2}} \neq 0,$$

they are all independent. We have, then, the following

THEOREMS.—(a) *The final equations of order unity are all independent.* Hence (b) *The final equations of all orders $\mu \geq 1$ are all independent.* Hence (c) *The secondary equations of all orders $\mu \geq 1$ are all independent.*

It may be noted in passing that the proofs in this section have not depended on any properties of the derivatives of the coefficients of the form nor on the coefficients themselves, except in that the discriminant a of the form shall not vanish.

§ 5. *Determination of an order μ for which all equations are independent.*

We shall now show that if $n \geq 3$ all the equations of order $\mu = 2$ are independent. We shall also obtain the result of ŻORAWSKI, that for $n = 2$ the equations of order $\mu = 2$ are not independent, but that the equations of order $\mu = 3$ are independent. It will be found possible, however, to simplify considerably ŻORAWSKI'S proof.

In this part of the work we can only show that the equations are independent for the general form. They may not be so for special forms. As we are seeking, however, the number of functions which are invariant when constructed from any form whatever, the dependence of the equations in the case of special forms is immaterial.

The method of proof is, as in the foregoing section, that of mathematical induction. If the equations of order two are not independent, then, whatever form the form ϕ may be, there can be found multipliers not all zero, such that when they are applied to the corresponding equations and the results added, they reduce to zero the coefficients of the variables p . We find the equations which these multipliers must satisfy. Then, if for any particular form ϕ there exist no multipliers, not identically zero, satisfying these equations, the independence is established. For the hypothesis was that such multipliers, not all zero, exist for every form. By the use of a certain particular type of form we are able to show that if the equations are not independent for the form in n variables they are not independent for the form in $n - 1$ variables. The reasoning may be applied again and again till the form in three variables is reached. We then show directly that the equations for the form in three variables are all independent. Hence it follows that the equations for the form in $n > 3$ variables are also independent.

As in the foregoing section we find that the introduction of new variables simplifies the work. We introduce the four-index symbols of CHRISTOFFEL,

$$(1) \quad (\lambda\mu, \nu\rho) \equiv \frac{1}{2} \left(a_{\lambda\rho} + a_{\mu\nu} - a_{\lambda\nu} - a_{\mu\rho} \right) \\ + \sum_{l=1}^n \sum_{m=1}^n A_{lm} \left(\left[\begin{smallmatrix} \lambda\rho \\ l \end{smallmatrix} \right] \left[\begin{smallmatrix} \mu\nu \\ m \end{smallmatrix} \right] - \left[\begin{smallmatrix} \lambda\nu \\ l \end{smallmatrix} \right] \left[\begin{smallmatrix} \mu\rho \\ m \end{smallmatrix} \right] \right),$$

and find that we have thereby partly solved the problem of determining the form of the invariants. For *

$$(2) \quad \beta_{\lambda\mu, \nu\rho} \equiv X(\lambda\mu, \nu\rho) \\ = - \sum_{r=1}^n \{ (r\mu, \nu\rho) \xi_{\lambda}^r + (\lambda r, \nu\rho) \xi_{\mu}^r + (\lambda\mu, r\rho) \xi_{\nu}^r + (\lambda\mu, \nu r) \xi_{\rho}^r \},$$

and this expression does not contain the second and third derivatives of the ξ 's. Hence the four-index symbols all satisfy the secondary equations of order two. But there are $N = n^2(n^2 - 1)/12$ independent four-index symbols,† and the secondary equations of order two can have at most N independent common solutions. For the number \bar{N}_{n^2} of variables in these equations, all of which are independent, is equal to the number of the first and second derivatives of the coefficients, a_{ik} , of the form; and the number \bar{M}_{n^2} of equations is equal to the number of second and third derivatives of the quantities ξ . The difference, $\bar{N}_{n^2} - \bar{M}_{n^2}$, gives the maximum number of independent common solutions of the equations. But

$$(3) \quad \bar{N}_{n^2} = \frac{n^2(n+1)(n+3)}{4}, \quad \bar{M}_{n^2} = \frac{n^2(n+1)(n+5)}{6},$$

therefore

$$(4) \quad \bar{N}_{n^2} - \bar{M}_{n^2} = \frac{n^2(n^2 - 1)}{12} = N.$$

Hence the invariants of order two can contain the first and second derivatives of the coefficients a_{ik} only through the N independent four-index symbols. Hence by the introduction of these symbols as new variables, the system of equations determining the invariants of order two is reduced to the system of n^2 equations obtained by introducing the N independent four-index symbols into the n^2 primary equations of order two.

These equations form a complete system. If relations exist among them an equal number of relations exist among the original equations. The problem, then, is reduced to that of determining the independence of these n^2 equations.

* Cf. LEVI-CIVITA, R. Ist. Veneto, Atti, ser. 7, vol. 5₂ (1894), p. 1454.

† CHRISTOFFEL, Crelle's Journal, vol. 70 (1869), p. 54.

The equations are obtained by equating to zero the coefficients of the n^2 quantities $\xi_{r,s}$ in the expression

$$(5) \quad Yf \equiv \sum_{i=1}^n \sum_{k=1}^i a_{ik} \frac{\partial f}{\partial a_{ik}} + \sum_{\lambda=1}^{n'} \sum_{\mu=1}^{\lambda-1} \sum_{\nu=1}^n \sum_{\rho=1}^{\nu-1} \beta_{\lambda\mu, \nu\rho} \frac{\partial f}{\partial (\lambda\mu, \nu\rho)}.$$

Here the accents on the signs \sum indicate that the summation extends only to those values of λ, μ, ν, ρ giving the independent symbols.

Introducing the values of $a_{ik}, \beta_{\lambda\mu, \nu\rho}$ already found we have

$$(6) \quad \begin{aligned} -Yf \equiv & \sum_{i=1}^n \sum_{k=1}^i \frac{\partial f}{\partial a_{ik}} \sum_{r=1}^n (a_{ir} \xi_r + a_{kr} \xi_r) \\ & + \sum_{\lambda=1}^{n'} \sum_{\mu=1}^{\lambda-1} \sum_{\nu=1}^n \sum_{\rho=1}^{\nu-1} \frac{\partial f}{\partial (\lambda\mu, \nu\rho)} \sum_{r=1}^n \{ (r\mu, \nu\rho) \xi_r + (\lambda r, \nu\rho) \xi_r \\ & + (\lambda\mu, r\rho) \xi_r + (\lambda\mu, \nu r) \xi_r \}, \end{aligned}$$

and we find that, as in the case of the final equations, the condition that the equations shall be dependent is the existence of a set of multipliers $\xi_{r,s}$, not all zero, satisfying the equations

$$(A) \quad -a_{ik} \equiv \sum_{r=1}^n \{ a_{ir} \xi_r + a_{kr} \xi_r \} = 0 \quad (i, k=1, 2, \dots, n; k \leq i),$$

$$(B) \quad \begin{aligned} -\beta_{\lambda\mu, \nu\rho} = & \sum_{r=1}^n (r\mu, \nu\rho) \xi_r + (\lambda r, \nu\rho) \xi_r + (\lambda\mu, r\rho) \xi_r + (\lambda\mu, \nu r) \xi_r = 0 \\ & (\lambda, \mu, \nu, \rho=1, 2, \dots, n; \mu < \lambda, \rho < \nu). \end{aligned}$$

The quantities $\xi_{r,s}$, if they exist, must be such functions of the quantities a_{ik} and of the four-index symbols that, if substituted in (A) and (B) they satisfy these equations identically. Thus if, for any single form, arbitrarily chosen save for the continuity of the coefficients and the non-vanishing of the discriminant, it happens that we find that there exist no n^2 quantities $\xi_{r,s}$, not all identically zero, satisfying these equations, then the equations determining the invariants of order two are all independent.

Consider, then, the special form,

$$\phi = \phi_1 + dx_n^2 \quad (a = 1 \cdot a_1 \neq 0),$$

where

$$\phi_1 = \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} a_{ik} (x_1 \cdots x_{n-1}) dx_i dx_k \quad (a_1 = |a_{11} \cdots a_{n-1, n-1}|),$$

Here ϕ_1 is a form in the $n-1$ variables x_1, \dots, x_{n-1} , only. It is easily seen that for such a special form ϕ ,

$$(7) \quad \begin{bmatrix} rn \\ s \end{bmatrix} \equiv 0, (nh, ik) \equiv 0 \quad (r, h, i, k=1, 2, \dots, n),$$

$$(8) \quad \begin{aligned} a_{rn} &= 0 & (r \neq n), \\ a_{nn} &= 1. \end{aligned}$$

Making use of these relations we find that equations (A) and (B) can be divided into sets as follows:

$$\text{I.} \quad -a_{ik} \equiv \sum_{r=1}^{n-1} \{a_{rk}\xi_r + a_{ri}\xi_k\} = 0 \quad (i, k=1, 2, \dots, n-1; k \leq i).$$

$$\text{II.} \quad -\beta_{\lambda\mu, \nu\rho} \equiv \sum_{r=1}^{n-1} \{ (r\mu, \nu\rho)\xi_r + (\lambda r, \nu\rho)\xi_r + (\lambda\mu, r\rho)\xi_r + (\lambda\mu, \nu r)\xi_r \} \\ (\lambda, \mu, \nu, \rho=1, 2, \dots, n-1; \mu > \lambda, \rho < \nu).$$

$$\text{III.} \quad -a_{nk} \equiv \sum_{r=1}^{n-1} \{a_{rk}\xi_r + a_{nn}\xi_k\} = 0 \quad (k=1, 2, \dots, n-1; a_{nn}=1).$$

$$\text{IV.} \quad -a_{nn} \equiv 2a_{nn}\xi_n = 0 \quad (a_{nn}=1 \neq 0).$$

$$\text{V.} \quad -\beta_{\lambda\mu, \nu\rho} \equiv \sum_{r=1}^{n-1} (\lambda\mu, \nu r)\xi_n = 0 \quad (\lambda, \mu, \nu \leq n-1).$$

$$\text{VI.} \quad -\beta_{\lambda\mu, \nu\rho} \equiv 0 \quad (\text{At least two indices, } \lambda, \mu, \nu, \rho \text{ are equal to } n).$$

We note that equations I, II are those which determine the multipliers for the equations satisfied by the invariants of order two for a form in $n-1$ variables.

Let us assume now that the equations which determine the invariants of order two for the general form in n variables are not all independent. Then they are, for the particular form ϕ , not all independent, whatever form in $n-1$ variables with non-vanishing discriminant the form ϕ_1 may be.

For such a form ϕ there exist, then, n^2 quantities, $\xi_{r,s}$, not all zero, satisfying equations I...VI.

Suppose now that the $(n-1)^2$ quantities $\xi_{r,s}$ ($r, s \equiv n-1$) are all identically zero. By IV, $\xi_{nn} \equiv 0$. Hence not all the $2(n-1)$ quantities

$$\xi_{1, n}, \dots, \xi_{n-1, 1, n}, \xi_{n_1, 1}, \dots, \xi_{n, n-1},$$

are identically zero. It follows from III that, since $a_{nn} = 1 \neq 0$, and the determinant of the quantities $\xi_{r,n}$ is $a_1 \neq 0$, the quantities $\xi_{1, n}, \dots, \xi_{n-1, n}$ all vanish if the quantities $\xi_{n, 1}, \dots, \xi_{n, n-1}$ all vanish, and conversely.

Hence for the forms under consideration not all the $n - 1$ quantities $\xi_{r, n}$ ($r = 1, 2, \dots, n - 1$) are identically zero. But they must satisfy equations V, and in particular, the following $n - 1$ of those equations:

$$(9) \quad -\beta_{1\mu, 1n} \equiv \xi_1 + \sum_{r=2}^{n-1} (1\mu, 1r) \xi_r = 0 \quad (\mu = 2, 3, \dots, n-1),$$

$$(10) \quad -\beta_{23, 2n} \equiv (23, 21) \xi_1 + 0 \xi_2 + \sum_{r=3}^{n-1} (23, 2r) \xi_r = 0.$$

Let R be the determinant of this system of equations. Then, since by hypothesis not all the quantities $\xi_{r, n}$ are identically zero, we must have

$$(11) \quad R \equiv 0.$$

But

$$(12) \quad R \equiv (23, 21) \begin{vmatrix} (12, 12), & (12, 13), & \dots, & (12, 1n-1) \\ (13, 12), & (13, 13), & \dots, & (13, 1n-1) \\ \vdots & \vdots & \ddots & \vdots \\ (1n-1, 12), & (1n-1, 13), & \dots, & (1n-1, 1n-1) \end{vmatrix}.$$

Now referring to the definition of the four-index symbols we easily see that we may take the form of ϕ_1 so that, at a given point,

$$(13) \quad \begin{aligned} (1\mu, 1r) &= (1r, 1\mu) = 0 & (r \neq \mu), \\ (1\mu, 1\mu) &= 1, \quad (23, 21) = 1. \end{aligned}$$

Then we have, at that point,

$$(14) \quad R = 1.$$

Hence

$$(15) \quad R \neq 0,$$

and we therefore have a contradiction.

It therefore follows that for all forms for which $R \neq 0$ all the $n - 1$ quantities $\xi_{r, n}$ ($r = 1, 2, \dots, n - 1$) vanish identically and hence, by III and IV, all the n quantities $\xi_{n, r}$ ($r = 1, 2, \dots, n$) must also vanish identically. But for all these forms the equations of order two are not all independent. Hence not all the $(n - 1)^2$ functions $\xi_{r, s}$ ($r, s \leq n - 1$) can be identically zero. But these quantities satisfy equations I and II; and we have therefore the result:

If the equations determining the invariants of order two are, for all forms in n variables, not independent, then there exist, for all forms in $n - 1$ variables which satisfy the relations

$$(16) \quad R \neq 0, \quad a_1 \neq 0,$$

$(n-1)^2$ quantities $\xi_{r,s}(r, s \leq n-1)$, not all identically zero, satisfying equations I and II.

Now the condition that equations I and II shall admit a set of solutions $\xi_{r,s}$, not all identically zero, is the vanishing of a certain number, p , of determinants formed from the matrix of their coefficients. The elements of these determinants are either coefficients $a_{ik}(i, k = 1, 2, \dots, n-1)$ of the form ϕ , or four-index symbols $(\lambda\mu, \nu\rho)$, $(\lambda, \mu, \nu, \rho \leq n-1)$. Moreover the condition is that all the p determinants vanish. They vanish for all values of the arguments a_{ik} , $(\lambda\mu, \nu\rho)$, which make

$$(17) \quad R \neq 0, \quad a_1 \neq 0,$$

and they are polynomials in those arguments. Let η be the number of arguments; ϵ , the highest degree to which any argument appears. Then we can certainly find, in the neighborhood of these values of the arguments which make

$$(18) \quad a_1 \neq 0, \quad R = 0,$$

$\epsilon + 1$ values of each of the η arguments such that, for any combination of these values,

$$(19) \quad a_1 \neq 0, \quad R \neq 0.$$

But for each of these combinations of the arguments there exist multipliers $\xi_{r,s}$, satisfying I and II and not all zero. Hence the p determinants in question vanish. But then, by a well-known theorem concerning polynomials, they vanish identically and hence, in particular, they vanish when $R = 0$. Hence the system of equations I and II admits a set of solutions, not all of which are zero, whether the form makes $R = 0$ or not. Hence the equations for the invariants of the second order for the form ϕ_1 are not all independent.

We have, therefore, the following

THEOREM.—*If the equations of order two are dependent for every form in n variables, they are dependent for every form in $n-1$ variables.*

COROLLARY.—*If the equations of order two are dependent for every form in $n > 3$ variables, they are dependent for every form in three variables.*

We shall now show that for the form in three variables the equations of order two are independent.

There are, in this case, six independent four-index symbols:

$$(20) \quad \begin{aligned} B_{11} &= (23, 23), & B_{12} &= (23, 31), & B_{13} &= (23, 12), \\ B_{22} &= (31, 31), & B_{23} &= (31, 12), \\ B_{33} &= (12, 12). \end{aligned}$$

To make the work more symmetrical we introduce, in place of the coefficients a_{ik} , the quantities, A_{ik} , obtained by dividing by a the co-factors of the elements a_{ik} in a ; and, in place of the four-index symbols B_{ik} , the quantities $C_{ik} = B_{ik}/a$.

If we put

$$(21) \quad P_{ik} = \frac{\partial f}{\partial A_{ik}}, \quad O_{ik} = \frac{\partial f}{\partial C_{ik}},$$

the equations of order two become:

$$(22) \quad \begin{aligned} 2A_{11}P_{11} + A_{12}P_{12} + A_{13}P_{13} + 2C_{11}O_{11} + C_{12}O_{12} + C_{13}O_{13} &= 0, \\ 2A_{21}P_{11} + A_{22}P_{12} + A_{23}P_{13} + 2C_{21}O_{11} + C_{22}O_{12} + C_{23}O_{13} &= 0, \\ 2A_{31}P_{11} + A_{32}P_{12} + A_{33}P_{13} + 2C_{31}O_{11} + C_{32}O_{12} + C_{33}O_{13} &= 0, \\ A_{11}P_{12} + 2A_{12}P_{22} + A_{13}P_{23} + C_{11}O_{12} + 2C_{12}O_{22} + C_{13}O_{23} &= 0, \\ A_{21}P_{12} + 2A_{22}P_{22} + A_{23}P_{23} + C_{21}O_{12} + 2C_{22}O_{22} + C_{23}O_{23} &= 0, \\ A_{31}P_{12} + 2A_{32}P_{22} + A_{33}P_{23} + C_{31}O_{12} + 2C_{32}O_{22} + C_{33}O_{23} &= 0, \\ A_{11}P_{13} + A_{12}P_{23} + 2A_{13}P_{33} + C_{11}O_{13} + C_{12}O_{23} + 2C_{13}O_{33} &= 0, \\ A_{21}P_{13} + A_{22}P_{23} + 2A_{23}P_{33} + C_{21}O_{13} + C_{22}O_{23} + 2C_{23}O_{33} &= 0, \\ A_{31}P_{13} + A_{32}P_{23} + 2A_{33}P_{33} + C_{31}O_{13} + C_{32}O_{23} + 2C_{33}O_{33} &= 0. \end{aligned}$$

The following determinant, formed from the first six and last three columns of the matrix of the system, does not vanish identically:

$$(23) \quad D \equiv \begin{vmatrix} 2A_{11} & A_{12} & A_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2A_{31} & A_{32} & A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{11} & 0 & 2A_{12} & A_{13} & 0 & 2C_{12} & C_{13} & 0 \\ 0 & A_{21} & 0 & 2A_{22} & A_{23} & 0 & 2C_{22} & C_{23} & 0 \\ 0 & A_{31} & 0 & 2A_{32} & A_{33} & 0 & 2C_{32} & C_{33} & 0 \\ 0 & 0 & A_{11} & 0 & A_{12} & 2A_{13} & 0 & C_{12} & 2C_{13} \\ 0 & 0 & A_{21} & 0 & A_{22} & 2A_{23} & 0 & C_{22} & 2C_{23} \\ 0 & 0 & A_{31} & 0 & A_{32} & 2A_{33} & 0 & C_{32} & 2C_{33} \end{vmatrix}$$

$$(24) \quad = 32 \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \left\{ \begin{vmatrix} A_{21} & A_{22} & A_{23} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} \begin{vmatrix} C_{31} & C_{32} & C_{33} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \right. \\ \left. - \begin{vmatrix} A_{31} & A_{32} & A_{33} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} \begin{vmatrix} C_{21} & C_{22} & C_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \right\}.$$

The equations of order two are, therefore, for the form in three variables, independent.*

From this we have, as a consequence, the

THEOREM.—*The equations of order two are all independent for the general form in $n \geq 3$ variables.*

Consequently, from this and the lemma of § 3 we have the

THEOREM.—*For the general form in $n \geq 3$ variables all the equations of order $\mu \geq 2$ are independent.*

Consider now the binary forms. Put

$$Z \equiv (12, 12),$$

$$(25) \quad Z_1 \equiv (12, 121) \equiv \frac{\partial(12, 12)}{\partial x_1} - (12, 12) \left\{ a_{22}a_{11} - 2a_{12}a_{11} + a_{11}a_{22} \right\},$$

$$Z_2 \equiv (12, 122) \equiv \frac{\partial(12, 12)}{\partial x_2} - (12, 12) \left\{ a_{22}a_{11} - 2a_{12}a_{12} + a_{11}a_{22} \right\}$$

$$q \equiv \frac{\partial f}{\partial Z}, \quad q_1 \equiv \frac{\partial f}{\partial Z_1}, \quad q_2 \equiv \frac{\partial f}{\partial Z_2}.$$

Then the equations of order two become

$$(26) \quad \begin{aligned} 2a_{11}p_{11} + a_{12}p_{12} &+ 2Zq = 0, \\ 2a_{21}p_{11} + a_{22}p_{12} &= 0, \\ a_{11}p_{12} + 2a_{12}p_{22} &= 0, \\ a_{21}p_{12} + 2a_{22}p_{22} + 2Zq &= 0. \end{aligned}$$

* Since the equations in twelve variables, which give the invariants of order two for the ternary form, are all independent, they have three common solutions, the three invariants of order two of the ternary form. CHRISTOFFEL pointed out that these may be obtained as algebraic simultaneous invariants of two quadratic forms (CHRISTOFFEL, Crelle's Journal, vol. 70 (1869), p. 65), and SOUVAROFF (Bulletin des Sciences Mathématiques et Astronomiques, vol. 4 (1873), p. 180) gave them in notation differing but slightly from that here used. The invariants are

$$H_1 = \frac{\{|C_{11}A_{22}A_{33}| + |A_{11}C_{22}A_{33}| + |A_{11}A_{22}C_{33}|\}}{|A_{11}A_{22}A_{33}|},$$

$$H_2 = \frac{\{|A_{11}C_{22}C_{33}| + |C_{11}A_{22}C_{33}| + |C_{11}C_{22}A_{33}|\}}{|A_{11}A_{22}A_{33}|},$$

$$H_3 = \frac{|C_{11}C_{22}C_{33}|}{|A_{11}A_{22}A_{33}|}.$$

The determinant of these equations is

$$(27) \quad \begin{vmatrix} 2a_{11} & a_{12} & 0 & 2Z \\ 2a_{21} & a_{22} & 0 & 0 \\ 0 & a_{11} & 2a_{12} & 0 \\ 0 & a_{21} & 2a_{22} & 2Z \end{vmatrix} \equiv 0.$$

Hence the equations are dependent. But as the discriminant does not vanish, not all third order minors of this determinant can vanish, and hence the equations can satisfy but one linear relation.

The equations of order three are

$$(28) \quad \begin{aligned} 2a_{11}p_{11} + a_{12}p_{12} &+ 2Zq + 3Z_1q_1 + 2Z_2q_2 = 0, \\ 2a_{21}p_{11} + a_{22}p_{12} &+ Z_2q_1 = 0, \\ a_{11}p_{12} + 2a_{12}p_{22} &+ Z_1q_2 = 0, \\ a_{21}p_{12} + 2a_{22}p_{22} + 2Zq + 2Z_1q_1 + 3Z_2q_2 &= 0. \end{aligned}$$

The following determinant of the matrix does not identically vanish:

$$\begin{vmatrix} a_{12} & 2Z & 3Z_1 & 2Z_2 \\ a_{22} & 0 & z_2 & 0 \\ a_{11} & 0 & 0 & z_1 \\ a_{21} & 2Z & 2Z_1 & 3Z_2 \end{vmatrix} = 2Z(a_{11}Z_2^2 - a_{22}Z_1^2).$$

Hence the equations are all independent.

We have therefore the following

THEOREM.—If $n \geq 3$, $\mu \geq 2$, or $n = 2$, $\mu \geq 3$, the equations determining the invariants of order μ are, for the general quadratic differential form in n variables, independent, and hence the number of such invariants is given by the excess of the number of the variables p in these equations, over the number of the equations themselves.

§ 6. Equations of orders zero and unity.

The equations of order zero may be written

$$(1) \quad \sum_{\lambda=1}^n (1 + \epsilon_{i\lambda}) a_{r\lambda} p_{i\lambda} = 0 \quad (r, i = 1, 2, \dots, n),$$

where

$$\epsilon_{i\lambda} = 0, \quad \text{if } i \neq \lambda; \quad \epsilon_{ii} = 1.$$

There are n^2 of these equations. They contain $n(n+1)/2$ variables $p_{i\lambda}$. It can readily be shown, by taking the n equations characterized by the index $i=1$, $n-1$ equations characterized by $i=2$, and so on, and finally, by taking one equation characterized by $i=n$, that we can form a set of $n(n+1)/2$ equations whose determinant does not vanish. This selection is made possible by the fact that since $a \neq 0$ not all the minors formed from any k columns can vanish identically. Hence we have the

THEOREM.—*The equations of order zero include as many independent equations as they have variables, and consequently have no common solutions.*

In passing to the equations of order unity, $n^2(n+1)/2$ new variables and the same number of equations, are added. The equations added are the final equations of order unity, and are, as we have seen, all independent. Hence, by the same reasoning as that of § 3, the system of $n^2 + n^2(n+1)/2$ equations in

$$\frac{n(n+1)}{2} + \frac{n^2(n+1)}{2} = N_{n1}$$

variables contains N_{n1} independent equations. We have, then, the

THEOREM: *The equations of order unity include as many independent equations as they have variables and hence have no common solutions.*

This is, of course, RICCI's result, but it is obtained in a wholly different way.

§ 7. Number of invariants.

The number of variables, $N_{n\mu}$, in the equations of order μ is the number of the quantities a_{ik} and of their derivatives of all orders not exceeding μ . The number of equations, $M_{n\mu}$, is the number of derivatives of the quantities ξ_r of all orders not exceeding $\mu+1$.

We have *

$$(1) \quad N_{n\mu} = \frac{n(n+1)}{2} \frac{(n+\mu)!}{\mu! n!},$$

$$(2) \quad M_{n\mu} = n \left\{ \frac{(n+\mu+1)!}{n! (\mu+1)!} - 1 \right\}.$$

The difference,

$$(3) \quad J_{n\mu} \equiv N_{n\mu} - M_{n\mu} = n + \left\{ \frac{(n-1)(\mu-1)-2}{2} \right\} \frac{(n+\mu)!}{(n-1)! (\mu+1)!},$$

is, if the equations of order μ are all independent, the number of invariants of order not greater than μ .

* Cf. LEVI-CIVITA, R. Ist. Veneto, Atti, ser. 7, vol. 52 (1894), p. 1466-8.

If the equations of order $\mu - 1$ are also independent there are $J_{n, \mu-1}$ invariants of order not greater than $\mu - 1$. Hence there are

$$(4) \quad I_{n\mu} \equiv J_{n\mu} - J_{n, \mu-1} = n \frac{\mu-1}{2} \frac{(n+\mu-1)!}{(n-2)!(\mu+1)!}$$

invariants of order μ .

Since neither the equations of order zero nor those of order unity have any common solutions, and since the equations of order two are, if $n > 2$, all independent, there are

$$(5) \quad J_{n2} = \frac{(n-2)(n-1)n(n+3)}{12}$$

invariants of order two.

Since for the binary form there are

$$(6) \quad J_{23} = 2$$

invariants of order not higher than three and since the four equations of order two contain four variables and include but three independent equations, one of these invariants is of order two and one of order three. For all other values of n and μ the equations of order $\mu - 1$ as well as those of order μ are all independent and hence there are $I_{n\mu}$ invariants.

We have, then, the following

THEOREM: *The number of invariants of order μ for the general quadratic form in n variables is*

$$I_{n\mu} = n \frac{\mu-1}{2} \frac{(n+\mu-1)!}{(n-2)!(\mu+1)!}$$

provided $n > 2$, $\mu > 2$, or $n = 2$, $\mu > 3$.

If $n = 2$ and $\mu = 2$ or $\mu = 3$, there is one invariant.

If $n > 2$ and $\mu = 2$, there are

$$J_{n2} = \frac{(n-2)(n-1)n(n+3)}{12}$$

invariants.

If $\mu = 0$ or $\mu = 1$, there are no invariants.

§ 8. Simultaneous invariants.

In studying those quadratic differential forms ϕ , in n variables, which can be expressed as the sum of $n + 1$ differential squares, it is found that to the form ϕ a second form ψ , also in n variables, can be adjoined, and the theory of the simultaneous invariants of these two forms is important in the study of the geo-

metric interpretation of ϕ as the squared linear element of an n -dimensional surface in a euclidean space of $n + 1$ dimensions. For $n = 2$, ϕ and ψ are the first and second fundamental forms of a surface, and the simultaneous invariant H , of order zero, is the mean curvature.

The invariants of order two of a ternary quadratic differential form can be expressed as the simultaneous invariants of order zero of two such forms.

The simultaneous invariants of two or more quadratic differential forms may be found by extending the group to the coefficients of those forms and to their derivatives. The equations determining the invariants may be obtained from those for a single form by annexing to them a set of terms constructed for each additional form in the same way as the terms already found for a single form. It is evident that if the equations are, for a single form, independent, they will be so for several forms.

From m forms we obtain then $m N_{n\mu}$ variables and $M_{n\mu}$ equations. Hence if the equations are, for a single form, all independent there will be

$$(1) \quad m N_{n\mu} - M_{n\mu}$$

invariants of order μ or lower. But of these

$$(2) \quad m(N_{n\mu} - M_{n\mu})$$

are invariants of the m separate forms. There remain

$$(3) \quad S_{mn\mu} = (m - 1) M_{n\mu}$$

true simultaneous invariants of the m forms. If the same conditions hold for the order $\mu - 1$ there will be

$$(4) \quad \begin{aligned} I_{mn\mu} &= S_{mn\mu} - S_{mn\mu-1} = (m - 1)(M_{n\mu} - M_{n\mu-1}) \\ &= (m - 1) \frac{(n + \mu)!}{(n - 1)!(\mu + 1)!} \end{aligned}$$

true simultaneous invariants of order μ for the m forms. If $m = 2$,

$$(5) \quad I_{2n\mu} = \frac{(n + \mu)!}{(n - 1)!(\mu + 1)!}.$$

Hence the $(m - 1)I_{2n\mu} = I_{2n\mu}$ true simultaneous invariants of m points may be taken as invariants of the $m - 1$ pairs of forms obtained by associating every given form with the remaining $m - 1$ in succession.

When the number of forms is greater than unity, invariants of order zero and unity also exist.

A detailed consideration of the special cases leads us to the following

THEOREM : *The number of true simultaneous invariants of order zero is, for m forms,*

$$m \frac{n(n+1)}{2} - n^2.$$

If $n = 2$ and $\mu = 3$, or $\mu = 2$ and $n > 2$, the number of true simultaneous invariants of order μ is, for m forms,

$$(m-1) \frac{n(n+\mu)!}{(\mu+1)!(n-1)!} + \frac{mn(n-1)}{2}.$$

For all other values of n and μ the number of true simultaneous invariants of order μ is

$$I_{mn\mu} = (m-1) \frac{n(n+\mu)!}{(\mu+1)!(n-1)!},$$

and the invariants can be taken as invariants of pairs of forms.

HARVARD UNIVERSITY,
June 1, 1901.
